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Fractional Lévy motions and related processes

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Abstract. This paper investigates some properties of a class of random motions called fractional Lévy motions (FLMs) and their fractal time extension.

FLM identifies with fractional Brownian motion (FBM) sampled in fractal Lévy time. This two parameter class of processes borrows hyperbolic temporal dependence to FBM and 'heavy-tailedness' to Lévy flights or motions (LM). It is shown that there exists a family of FLMs which shares with standard Brownian motion (BM) the (strict) diffusivity property that the dispersion (measured in terms of quantiles) grows as time raised to the power $\frac{1}{2}$. Processes from this class are critical in that they separate both sub/superdiffusive FLMs and finite/infinite-variance motions. Related Ornstein–Uhlenbeck and multiplicative processes are also briefly investigated. FLMs are self-similar but not Lévy-stable, in sharp contrast to Brownian (whether fractional or not) and standard or fractional stable (in the Taqqu–Wolpert sense) Lévy motions (FSMs). Stable processes are self-similar as a result of Lévy stability. However, the converse is false; there are motions with stationary increments which are self-similar but not stable: the fractional Lévy motion is one of them, in the pure jump process class. It turns out that all processes discussed so far (BM, FBM, LM, FLM, FSM) are self-similar with stationary increments.

We finally introduce a natural one-parameter δ-family of 'fractional' processes for which a weaker notion of self-similarity seems to hold, i.e. self-similarity of the unidimensional distributions. It is fractal time Brownian motion (FT-BM). Such a process is obtained as a weak limit of fractal time random walk models, with $\delta \in (0, 1)$ the tail exponent of the waiting times; FT-BM identifies with Brownian motion now sampled in fractal inverse Lévy time. This construction extends to FBM, LM and FLM: we therefore introduce and study FT-FBM, FT-LM and FT-FLM which are the fractal time extensions of FBM, LM and FLM. These processes are not strongly self-similar, nor stable, nor do they have stationary increments.

1. Introduction

Standard Brownian motion (BM) is a Gaussian non-stationary process. It has stationary independent increments (SIIs). Its trajectories are continuous but nowhere differentiable; it is self-similar (SS) with exponent $\frac{1}{2}$. Let $\overline{B}_{1/2}(t)$ be such a process at time t, with $\overline{B}_{1/2}(0) = 0$. Brownian motion is well known to be the microscopic model for heat diffusion.

To explain certain statistical time series arising from finance and hydrology (which exhibit the so-called Hurst phenomenon [13, 26, 27] with exponent $H \neq \frac{1}{2}$) a stochastic motion of a completely different nature was designed by Mandelbrot and Van Ness in [22]: so-called fractional BM (FBM) with exponent $H \in (0, 1)$, say $\overline{B}_H(t)$. Chiefly, the hypothesis that the increments of the previous motion were independent was released while maintaining all other properties of standard BM: in particular, continuity of the sample paths and stability under addition (which here means Gaussianity). As a result, the essential feature of FBM is that the increment process $B_H(t) := \overline{B}_H(t+1) - \overline{B}_H(t)$ over (say) a unit period of time, while stationary, presents the property that the covariance function grows as

$$\boldsymbol{E}[B_H(s)B_H(s+t)] \underset{t\uparrow+\infty}{\sim} t^{2(H-1)} \tag{1.1}$$

i.e. has the *power-law decay* property if $H \neq \frac{1}{2}$. This is in sharp contrast to other 'standard' stationary processes (such as Ornstein–Uhlenbeck's: $\sigma \mathrm{e}^{-at/2}\overline{B}_{1/2}(\mathrm{e}^{at})$, $a, \sigma > 0$) whose covariance function has exponential decay $\sigma^2 \exp -a|t|$. As a result, the spectral density of $B_H(t)$ behaves as $1/|f|^{2H-1}$ in the vicinity of the frequency $f = 0$, showing the divergence syndrome as soon as $H > \frac{1}{2}$. This is the signature that 'long cycles' are present in the phenomenon to be modelled by FBM which could be one of the possible explanations of the Hurst phenomenon (other explanations exist: see, e.g., [4]). In addition, FBM can be shown to be SS with exponent H which means that, with $a > 0$,

$$\{\overline{B}_H(at)\}_{t\in\mathbb{R}} \overset{d}{=} \{a^H \overline{B}_H(t)\}_{t\in\mathbb{R}}. \tag{1.2}$$

Here, symbol $\overset{d}{=}$ indicates that the two processes share the same finite-dimensional probability distributions.

Another important feature of FBM concerns 'anomalous diffusion' since it can easily be shown that the standard deviation (a dispersion measure) of FBM grows as

$$\boldsymbol{\sigma}[\overline{B}_H(t)] \propto t^H. \tag{1.3}$$

This means that $\overline{B}_H(t)$ is subdiffusive or antipersistent if $H < \frac{1}{2}$, while it is superdiffusive or persistent if $H > \frac{1}{2}$ (a model of enhanced diffusion). As a result, the trajectories of $\overline{B}_H(t)$ are less (more) 'regular' than those of BM in the case $H < \frac{1}{2}$ ($H > \frac{1}{2}$) and H is indeed a measure of this regularity: the Hurst–Hölder exponent of the trajectories. In more precise terms, $2 - H$ is the Hausdorff dimension of the space–time graph of $\overline{B}_H(t)$.

In section 2, we briefly recall some known facts about FBM that will be useful in what follows.

When dealing with anomalous diffusion, another class of processes of interest can be obtained from the idea of standard BM by loosening the Gaussian hypothesis; in this extension, the stability under addition requirement is, however, maintained. Searching for stable symmetric processes with SII that are SS yields the so-called symmetric α-stable (sαs) class of motions, also called Lévy motions (LMs) or Lévy flights, $\overline{B}^\alpha(t)$. Here $\alpha \in (0, 2)$ and the self-similarity exponent of $\overline{B}^\alpha(t)$ is $1/\alpha > \frac{1}{2}$. These random motions have no continuous sample paths (very large jumps are present but also tiny ones); one of the by-products of these large jumps is that the mean and variance of $\overline{B}^\alpha(t)$ are ill defined (indefinite and infinite); however, other equivalent measures of their 'dispersion' in time, $\boldsymbol{D}(\overline{B}^\alpha(t))$, show that

$$\boldsymbol{D}(\overline{B}^\alpha(t)) \underset{t\uparrow+\infty}{\propto} t^{1/\alpha}. \tag{1.4}$$

As $1/\alpha > \frac{1}{2}$, this underlines their superdiffusive character, chiefly arising from the 'heavy-tail' character of $\overline{B}^\alpha(t)$. In more precise terms

$$\boldsymbol{P}(|\overline{B}^\alpha(t)| > x) \underset{x\uparrow+\infty}{\propto} t/x^\alpha \tag{1.5}$$

suggesting power-law tails with tail index α. As $\alpha \in (0, 2)$, this property shows that LMs necessarily have infinite variance. This class of processes is also revisited in section 3 to make things self-coherent and, it is hoped, to add some new insight here and there.

In section 4, we are interested in a motion, $\overline{B}^\alpha_H(t)$, which presents *both* remarkable characters of the two processes just described: power-law decay of the covariance function (*hyperbolic dependence*) and heavy tails of the power-law type of the increment process, but not necessarily with infinite variance. We call it the fractional Lévy motion (FLM). FLM may be basically seen as FBM in fractal Lévy time: the trajectories of the FLM are discontinuous. From this definition, it appears that this process is no longer stable under addition, whereas it is SS with SS exponent $2H/\alpha > 0$, and with stationary increments (SIs). In addition, for some measure of dispersion

$$\boldsymbol{D}(\overline{B}_H^\alpha(t)) \underset{t\uparrow+\infty}{\propto} t^{2H/\alpha}. \tag{1.6}$$

Subdiffusivity (superdiffusivity) is obtained if $2H/\alpha < \frac{1}{2}$ (respectively $2H/\alpha > \frac{1}{2}$), so that strict diffusivity is achieved in the critical domain defined by $2H/\alpha = \frac{1}{2}$. This shows that their exists a one-parameter family $\overline{B}_{\alpha/4}^\alpha(t)$, $\alpha \in (0, 2)$ with diffusivity properties comparable to those of the standard BM but, of course, of completely different statistical nature (mixing power-law decay of the covariance function and hyperbolic tail behaviour). The central process of this family is $\overline{B}_{1/4}^1(t)$ is obtained with $\alpha = 1$ which one may call the fractional Cauchy process.

For this class of processes, some control of the tail behaviour holds. In more precise terms, it may be shown that

$$\boldsymbol{P}(|\overline{B}_H^\alpha(t)| > x) \underset{x\uparrow+\infty}{\propto} t/x^{\alpha/(2H)}. \tag{1.7}$$

The tail index is now $\alpha/(2H)$, which is allowed to vary on the whole positive real line, as $\alpha \in (0, 2)$ and $H \in (0, 1)$. Thus the critical domain $2H/\alpha = \frac{1}{2}$ also separates situations with finite ($(2H)/\alpha < \frac{1}{2}$) or infinite ($(2H)/\alpha > \frac{1}{2}$) variance.

In section 5, related interesting stationary processes are briefly introduced: the generalized Ornstein–Uhlenbeck class and multiplicative FLM processes.

BM and LM have SIIs and are stable (either Gaussian- or α-Lévy-stable). Loosening the condition that the increments are independent, FBM and fractional stable motion (FSM), as defined in [38], still have SIs and remain stable; the increment processes now exhibit interesting (hyperbolic) temporal dependence.

Processes with SIs and stable are SS: some statistical space–time invariance under appropriate dilation holds. This property is an important issue in physics [1, 26].

The definition of FLM shows that there are interesting processes with hyperbolic dependence which are SS-SI but not stable, with a richer tail behaviour than FSM (in the sense that finite variance is allowed).

Regardless of the stability property, all the processes discussed so far (BM, FBM, LM, FLM, FSM) are SS *and* with SIs (SS-SIs).

In section 6, we present a construction of a natural one-parameter δ-family of 'fractional' processes for which a weaker notion of self-similarity seems to hold, i.e. self-similarity of the unidimensional distributions: this is fractal time BM (FT-BM), ${}^\delta\overline{B}(t)$. Such a process is obtained as a weak limit of fractal time random walk models, with $\delta \in (0, 1)$ the tail exponent of the waiting times. FT-BM identifies with BM sampled in fractal inverse Lévy time. This construction extends to FBM, LM and FLM: we therefore introduce and study FT-FBM, FT-LM and FT-FLM which are the fractal time versions of FBM, LM and FLM. We denote by the FT-FLM process ${}^\delta\overline{B}_H^\alpha(t)$. For some measure of dispersion, we show that

$$\boldsymbol{D}[{}^\delta\overline{B}_H^\alpha(t)] \underset{t\uparrow+\infty}{\propto} t^{(2\delta H)/\alpha}.$$

Subdiffusivity (superdiffusivity) of FT-FLM is now obtained if $(2\delta H)/\alpha < \frac{1}{2}$ (respectively $(2\delta H)/\alpha > \frac{1}{2}$), so that strict diffusivity is achieved in the critical domain defined by $(2\delta H)/\alpha = \frac{1}{2}$.

This construction leads to increment processes which are *not* stationary. As a result, such FT processes are neither SS in the strong sense, nor stable, nor do they have SIs.

2. FBM revisited

2.1. Standard results

With $(x)_+ := \max(0, x)$, the FBM can be defined [22] via the Wiener (moving average) integral ($H \neq \frac{1}{2}$):

$$\overline{B}_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} ((t - s)_+^{H-1/2} - (-s)_+^{H-1/2})\, \mathrm{d}\overline{B}_{1/2}(s) \tag{2.1}$$

with respect to BM $\overline{B}_{1/2}(t)$ on the real line with $\overline{B}_{1/2}(0) = 0$. This is the single Gaussian (hence stable) process [23–25] with SIs with the self-similarity property (1.2). By the property of stationarity of the increments,

$$\{\overline{B}_H(t + s) - \overline{B}_H(t)\}_{t\in\mathbb{R}} \overset{d}{=} \{\overline{B}_H(s) - \overline{B}_H(0)\}_{t\in\mathbb{R}} \qquad \text{for all} \quad s \in \mathbb{R}. \tag{2.2}$$

Just like standard BM, this Gaussian process has zero mean but is not stationary. Indeed, from the stationarity of variances, its covariance is easily shown to satisfy, using (1.3),

$$\boldsymbol{E}[\overline{B}_H(s)\overline{B}_H(t)] \propto |s|^{2H} + |t|^{2H} - |t - s|^{2H}. \tag{2.3}$$

By contrast, the increment process $B_H(t)$ is Gaussian and stationary (in the strong and weak sense); its covariance function may be computed from (2.3) to give, if the chosen period is one,

$$\boldsymbol{E}[B_H(s)B_H(s + t)] \propto (|t + 1|^{2H} + |t - 1|^{2H} - 2|t|^{2H}) \tag{2.4}$$

thus independent of time s and with hyperbolic fading $t^{2(H-1)}$ as $t \uparrow +\infty$. The spectral density $S(f)$ (as the Fourier transform of the covariance function) is, with f the (frequency) Fourier variable,

$$S(f) \propto (1 - \cos(2\pi f))|f|^{-(2H+1)}. \tag{2.5}$$

Thus

$$S(f) \underset{f\uparrow 0}{\sim} 1/|f|^{2H-1} \qquad \text{and} \qquad S(f) \underset{f\uparrow +\infty}{\sim} 1/|f|^{2H+1}. \tag{2.6}$$

This shows that if $H > \frac{1}{2}$, $S(f)$ diverges at zero. Cycles with very long period are at stake in $B_H(t)$: this is the *long-range dependence* property; if $H < \frac{1}{2}$, although the covariance function still presents power-law decay (the *hyperbolic dependence* property), 'long cycles' are rare as $S(f) \underset{f\uparrow 0}{\rightarrow} 0$; on the other hand, cycles with short period are more numerous than in the case $H > \frac{1}{2}$. This is the so-called *negative-dependence* property. In the limit $H \uparrow 0^+$, the spectral density $S(f) \underset{f\uparrow +\infty}{\sim} 1/|f|^{1^+}$ and we get the celebrated $1/f$ noise [32]. This is a hint that trajectories with $H < \frac{1}{2}$ are more tortuous than those with $H > \frac{1}{2}$: the parameter H is the Hurst–Hölder measure of the regularity of the sample paths of $\overline{B}_H(t)$. Note that H is the key parameter in the understanding of *both* low- and high-frequency behaviour of $B_H(t)$. It may be shown that the (Hausdorff) dimension of the trail (respectively the space–time graph) of $\overline{B}_H(t)$ is $\min(1, 1/H) = 1$ (respectively $2 - H$) and that $\overline{B}_H(t)$ is $1/H$-variation bounded.

Remark 1. The fact that $\overline{B}_H(t)$ is H-SS, Gaussian with variance t^{2H} is *not* characteristic of this process: the stationarity of the increments is also a central point. To see this, consider the process $\overline{B}_{1/2}(t^{2H})$ which is standard BM in deterministic 'local time' t^{2H}, different from 'clock time' t. (This operation is merely a deterministic 'change of time' which will be allowed to be *random* in section 4, leading to very distinct features.) This process is easily seen to be H-SS, Gaussian with variance t^{2H}. However, it has *no* SIs which may be deduced from its covariance function which is $\inf(s^{2H}, t^{2H})$, in sharp contrast with that (2.3) of FBM.

Remark 2. There are some current projects [3, 30, 34] (and see also [12] for a work in this direction in a deterministic setup and for the bridge process), whose ambition is to design random motions similar to FBM, excepting that the Hurst–Hölder coefficient is allowed to vary with time. Depending on the type of variability of $H(\cdot)$ with time, these more general processes are called multifractional BM (MBM) if function $H(\cdot)$ is Hölder continuous or even generalized MBM (GMBM) allowing for discontinuities in $H(\cdot)$. The central feature here is a loosening of the condition that the increments of the FBM are stationary but we shall not enter into this interesting problem in this paper.

3. Lévy flights revisited

We review in this section some basic ingredients characterizing LMs [10,20,35]. For examples of applications in physics see [1, 5, 14, 16, 18, 36, 37] and references therein.

3.1. Most remarkable statistical properties

With $\alpha \in (0, 2)$, sαs LMs are processes with stationary independent increments. They share the stability property

$$\forall t > 0, \forall c \in (0,1) : \{c^{1/\alpha}\overline{B}_1^{\alpha}(t) + (1-c)^{1/\alpha}\overline{B}_2^{\alpha}(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{\overline{B}^{\alpha}(t)\}_{t\in\mathbb{R}} \qquad (3.1)$$

where $(\overline{B}_1^{\alpha}(t), \overline{B}_2^{\alpha}(t))$ are two independent statistical copies of $\overline{B}^{\alpha}(t)$. This notation is, as usual, relative to any finite-dimensional distributions of $\overline{B}^{\alpha}(t)$. The stability property is a particular case of a more general one, called semi-stability, introduced in [19], allowing for shifts. It may be shown from (3.1) that the following self-similarity property holds:

$$\{\overline{B}^{\alpha}(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{a^{1/\alpha}\overline{B}^{\alpha}(t)\}_{t\in\mathbb{R}} \qquad a > 0. \qquad (3.2)$$

α-stable LMs are SS with self-similarity index $1/\alpha$.

As a result, the characteristic function of the unidimensional distribution reads

$$\overline{\Phi}^{\alpha}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}^{\alpha}(t)} := \mathrm{e}^{-t|\lambda|^{\alpha}} \qquad \lambda \in \mathbb{R} \quad \alpha \in (0,2) \qquad (3.3)$$

which is the Lévy–Khintchine representation of such processes [21]. Here λ is the Fourier variable, dual of space.

Characteristic functions defined by (3.3) are real with respect to λ which indicates that the underlying probability density function (DF) of $\overline{B}^{\alpha}(t)$, say $f^{\alpha}(t, x)$, is symmetric in space x: $f^{\alpha}(t, x) = f^{\alpha}(t, -x)$. The expression of this density is known from the work of Bergström Feller [8] and has the following formal expression:

$$f^{\alpha}(t,x) = \frac{1}{t^{1/\alpha}} f^{\alpha}(x/t^{1/\alpha}) \qquad \text{with} \quad f^{\alpha}(x) = \frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{-y^{\alpha}} \cos(xy)\,\mathrm{d}y \qquad (3.4)$$

which, in the case $\alpha = 1$ (Cauchy) reduces to the well known closed form

$$f^{\alpha=1}(t,x) = \frac{1}{\pi t}/(1 + (x/t)^2). \qquad (3.5)$$

Now, if $F^{\alpha}(t,x) := \boldsymbol{P}(\overline{B}^{\alpha}(t) \leqslant x)$ is the probability distribution function (PDF) of $\overline{B}^{\alpha}(t)$, the symmetry property of the DF translates into $F^{\alpha}(t,x) = \overline{F}^{\alpha}(t,-x)$, with $\overline{F}^{\alpha}(t,x) := 1 - F^{\alpha}(t,x)$ the complementary PDF. In addition, it is known, using (3.3), that

$$\overline{F}^{\alpha}(t,x) \propto t x^{-\alpha} \qquad \text{as} \quad x \uparrow +\infty. \tag{3.6}$$

Thus, $\overline{B}^{\alpha}(t)$ is heavy tailed with tail index α (the Zipf–Pareto model [33, 44]).

From the expression of the characteristic function in (3.3), one also concludes that $\overline{B}^{\alpha}(t)$ has no moment of any order; in particular, the mean $\boldsymbol{E}[\overline{B}^{\alpha}(t)]$ is indefinite, whereas the order-two moment $\boldsymbol{E}[(\overline{B}^{\alpha}(t))^2]$ is infinite: moments are inappropriate in the Lévy variables context.

However, a natural measure of centrality in such a context is the median function $x^{\alpha}(t)$ defined by $F^{\alpha}(t, x^{\alpha}(t)) = \frac{1}{2}$ which is the null function here from the symmetry property. Additionally, a good measure of dispersion is the 'most probable fluctuation' function, $d^{\alpha}(t)$, defined here by

$$\boldsymbol{P}(|\overline{B}^{\alpha}(t) - x^{\alpha}(t)| > d^{\alpha}(t)) = \tfrac{1}{2} \tag{3.7}$$

in terms of quantiles. From $x^{\alpha}(t) = 0$ and the symmetry property of the PDF, $d^{\alpha}(t)$ is defined by $\overline{F}^{\alpha}(t, -d^{\alpha}(t)) - \overline{F}^{\alpha}(t, d^{\alpha}(t)) = \frac{1}{2}$, hence by $\overline{F}^{\alpha}(t, d^{\alpha}(t)) = \frac{1}{4}$. Consequently, from the tail equivalence, the asymptotic form of the dispersion of $\overline{B}^{\alpha}(t)$, $\boldsymbol{D}(\overline{B}^{\alpha}(t)) := d^{\alpha}(t)$, is

$$\boldsymbol{D}(\overline{B}^{\alpha}(t)) \underset{t\uparrow+\infty}{\propto} t^{1/\alpha}. \tag{3.8}$$

As $\alpha \in (0, 2)$, $1/\alpha > \frac{1}{2}$; this shows that, with the dispersion function measured in terms of quantiles, $\overline{B}^{\alpha}(t)$ is *always* superdiffusive. This result follows from the heavy-tailed character of the jumps generating this process, and from the infinite-variance property.

Let us here make a remark underlining the central role played by this measure of dispersion.

Remark 3. These measures of centrality and dispersion are also meaningful in the Gaussian FBM and BM setup. The FBM has the Gaussian density

$$f_H(t,x) = \frac{1}{\sqrt{2\pi}\, t^H} \exp\left[-\frac{x^2}{2t^{2H}}\right]. \tag{3.9}$$

Letting $F_H(t,x) := \boldsymbol{P}(\overline{B}_H(t) \leqslant x)$ stand for the PDF of $\overline{B}_H(t)$, the median function of $\overline{B}_H(t)$ is zero for symmetry reasons and its most probable fluctuation, $d_H(t)$, is defined by $\overline{F}_H(t, d_H(t)) = \frac{1}{4}$. Now, from l'Hospital's rule it is well known that

$$\overline{F}_H(t,x) \sim \frac{t^H}{x} \exp\left[-\frac{x^2}{2t^{2H}}\right] \qquad \text{for large} \quad x. \tag{3.10}$$

This shows that $\boldsymbol{D}(\overline{B}_H(t)) := d_H(t)$, verifies the asymptotics

$$\boldsymbol{D}(\overline{B}_H(t)) \underset{t\uparrow+\infty}{\propto} t^H \tag{3.11}$$

to be compared with (1.3) which, however, is exact.

Thus, the common dispersion measure to be used when comparing the spreading of $\overline{B}_H(t)$ and $\overline{B}^{\alpha}(t)$ is the most probable fluctuation.

We finally underline an interesting additional property of LM concerning large deviation. The tail behaviour (3.6) allows us to derive some additional properties of the LM. Let

$$\overline{B}^{\alpha}_{*}(t) := \sup_{s\leqslant t} \overline{B}^{\alpha}(t) \tag{3.12}$$

be the supremum LM. From the property that LM has SII, it follows from elementary calculus on extreme statistics [9, 11] that

$$\boldsymbol{P}(\overline{B}_*^{\alpha}(t) > x) = 1 - (1 - \boldsymbol{P}(\overline{B}^{\alpha}(1) > x))^t \tag{3.13}$$

so that, using (3.6),

$$\boldsymbol{P}(\overline{B}_*^{\alpha}(t) > x) \underset{x\uparrow+\infty}{\propto} t x^{-\alpha}. \tag{3.14}$$

Thus $\overline{B}^{\alpha}(t)$ and $\overline{B}_*^{\alpha}(t)$ are tail-equivalent which means [7] that

$$\frac{\boldsymbol{P}(\overline{B}^{\alpha}(t) > x)}{\boldsymbol{P}(\overline{B}_*^{\alpha}(t) > x)} \underset{x\uparrow+\infty}{\to} 1. \tag{3.15}$$

A consequence of (3.14), (3.15) is the following: if $\alpha > 1$, we have, from the law of large numbers $\frac{1}{t}|\overline{B}^{\alpha}(t)| \to v_0 := \boldsymbol{E}|\overline{B}^{\alpha}(1)| < +\infty$. Letting $|x| = tv$, $v > v_0$, formula (3.6) is also

$$\boldsymbol{P}\left(\frac{1}{t}|\overline{B}^{\alpha}(t)| > v\right) \underset{t\uparrow+\infty}{\propto} t^{1-\alpha} v^{-\alpha}. \tag{3.16}$$

This large deviation result suggests that, if $\alpha > 1$, the probability that the space–time graph of $\overline{B}^{\alpha}(t)$ quits the cone of equation $|x| = vt$ tends to zero as t raised to the power $1 - \alpha$. By contrast, if $\alpha < 1$, the probability that the space–time graph of $\overline{B}^{\alpha}(t)$ remains in the cone of equation $|x| = vt$ tends to zero as $\exp(-t^{1-\alpha})$:

$$\boldsymbol{P}\left(\frac{1}{t}|\overline{B}^{\alpha}(t)| \leqslant v\right) \underset{t\uparrow+\infty}{\propto} \exp(-t^{1-\alpha} v^{-\alpha}). \tag{3.17}$$

In the first case, $\alpha > 1$, the amplitudes of the (heavy-tailed) jumps in the LM are not strong enough to allow for a departure from a 'conic' diffusion as time drifts to infinity, whereas if $\alpha < 1$ they are indeed.

3.2. LM is BM in fractal Lévy time

We now recall that $\overline{B}^{\alpha}(t)$ is simply BM $\overline{B}_{1/2}(s)$ in fractal time $s = \overline{S}^{\alpha/2}(t)$, which therefore acts as a 'subordinator' (a notion due to S Bochner and worked out by W Feller, S J Taylor and W E Pruitt: see [26] for the bibliography and [6, 29] for different subordinators such as log-normal). This simple observation is the central point in the definition of fractional LM in the next section.

Let $\overline{S}^{\alpha/2}(t) \geqslant 0$ be an increasing process with SII whose Laplace transform is

$$\boldsymbol{E}\mathrm{e}^{-p\overline{S}^{\alpha/2}(t)} = \mathrm{e}^{-tp^{\alpha/2}} \qquad p \geqslant 0. \tag{3.18}$$

The random variable $\overline{S}^{\alpha/2}(t)$ admits the density at point $s > 0$

$$g^{\alpha/2}(t, s) = \sum_{n\geqslant 1} \frac{(-1)^n}{n!} t^n \frac{1}{\Gamma(-n\alpha/2)} s^{-(n\alpha/2+1)}. \tag{3.19}$$

This density satisfies the identity

$$g^{\alpha/2}(t, s) = \frac{1}{t^{2/\alpha}} g^{\alpha/2}(s/t^{2/\alpha}) \qquad \text{with} \quad g^{\alpha/2}(s) := g^{\alpha/2}(1, s). \tag{3.20}$$

If $\alpha = 1$, this expression reduces to

$$g^{1/2}(t, s) = \frac{1}{\sqrt{2\pi}} \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/2s}. \tag{3.21}$$

It can be shown that such processes are asymmetric Lévy processes (called subordinators) because the jumps of the increments are positive with unbounded Lévy measure with density

$$\frac{\alpha}{2\Gamma(1-\alpha/2)} \cdot x^{-(\alpha/2+1)} \qquad \text{for} \quad x > 0. \tag{3.22}$$

The sample paths of $\overline{S}^{\alpha/2}(t)$ exhibit (positive) jumps with very small amplitude together with very large hyperbolic jumps.

Next, if $\overline{B}_{1/2}(t)$ and $\overline{S}^{\alpha/2}(t)$ are independent,

$$\overline{B}^{\alpha}(t) \stackrel{d}{=} \overline{B}_{1/2}(\overline{S}^{\alpha/2}(t)). \tag{3.23}$$

Indeed, if this is the case and if $g^{\alpha/2}(t,s)$ is the density of $\overline{S}^{\alpha/2}(t)$ at s, Bayes' formula yields

$$\boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}^{\alpha}(t)} = \int_0^{+\infty} \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}_{1/2}(s)} g^{\alpha/2}(t,s)\,\mathrm{d}s = \int_0^{+\infty} \mathrm{e}^{-s\lambda^2/2} g^{\alpha/2}(t,s)\,\mathrm{d}s = \mathrm{e}^{-t(\frac{\lambda^2}{2})^{\alpha/2}} = \mathrm{e}^{-\frac{t}{2^{\alpha/2}}|\lambda|^{\alpha}}. \tag{3.24}$$

This operation is simply a change of time, so that $\overline{B}^{\alpha}(t)$ can be interpreted as BM $\overline{B}_{1/2}(s)$ in fractal (Lévy-stable) time $s = \overline{S}^{\alpha/2}(t)$: in other words $\overline{B}^{\alpha}(t)$ is a subordinate of BM. For example, if $\alpha = 1$, the density of the random variable $\overline{B}^{1}(t)$ (Cauchy motion) is easily shown to be, from (3.21),

$$f^1(t,x) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi s}} \mathrm{e}^{-x^2/2s} \frac{1}{\sqrt{2\pi}} \frac{t}{s^{3/2}} \mathrm{e}^{-t^2/2s}\,\mathrm{d}s = \frac{t}{\pi}\frac{1}{t^2+x^2} \tag{3.25}$$

as expected from (3.5).

4. Fractional LMs

We now define symmetric fractional LM (FLM), $\overline{B}_H^{\alpha}(t)$, in the following way:

$$\overline{B}_H^{\alpha}(t) := \overline{B}_H(\overline{S}^{\alpha/2}(t)) \tag{4.1}$$

thus as FBM in independent Lévy time. This approach is a natural extension to FBM of definition (3.23) which only concerned BM, and was suggested in [26, p 292]. The definition (4.1) and the moving average representation (2.1) of FBM yields a moving average representation of FLM and of the increment process, with standard BM as driving integrator.

Let us now exhibit some properties of this process. This process is SS, with SS index $2H/\alpha > 0$:

$$\{\overline{B}_H^{\alpha}(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{a^{(2H)/\alpha}\overline{B}_H^{\alpha}(t)\}_{t\in\mathbb{R}} \qquad a > 0 \tag{4.2}$$

as a result of the SS properties of its constitutive processes $\{\overline{S}^{\alpha/2}(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{a^{2/\alpha}\overline{S}^{\alpha/2}(t)\}_{t\in\mathbb{R}}$ and $\{\overline{B}_H(at)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{a^H\overline{B}_H(t)\}_{t\in\mathbb{R}}$. It has SIs because this property is shared by $\overline{B}_H(s)$ and $\overline{S}^{\alpha/2}(t)$. It has discontinuous sample path (as a subordinate of an asymmetric Lévy process). In some sense (see, e.g., [26, p 292]), the trail of $\overline{B}_H^{\alpha}(t)$ has intrinsic dimension $\min(1, \alpha/(2H))$. However, the process $\overline{B}_H^{\alpha}(t)$ is *not* stable. Indeed, its characteristic function is

$$\overline{\Phi}_H^{\alpha}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}_H^{\alpha}(t)} = \int_0^{+\infty} \mathrm{e}^{-s^{2H}\lambda^2/2} g^{\alpha/2}(t,s)\,\mathrm{d}s \tag{4.3}$$

which has nothing to do with the characteristic function of a stable distribution. It involves the Laplace transform, say $\psi_{\alpha,H}(x)$, of the variable $S_{\alpha/2}^{2H}$ with $S_{\alpha/2} := \overline{S}^{\alpha/2}(1)$ a Lévy-stable variable, i.e. $S_{\alpha/2}$ raised to the power $2H$. Indeed, from (3.20), formula (4.3) reduces to

$$\overline{\Phi}_H^{\alpha}(t,\lambda) = \psi_{\alpha,H}(t^{(4H)/\alpha}\lambda^2/2) \qquad \text{with} \quad \psi_{\alpha,H}(x) := \int_0^{+\infty} \mathrm{e}^{-xs^{2H}} g^{\alpha/2}(s)\,\mathrm{d}s \quad x \geqslant 0. \tag{4.4}$$

Using (3.19), (3.20), we get the power-series expression

$$\psi_{\alpha,H}(x) = 1 + \sum_{n\geqslant 1} \frac{(-1)^n}{n!} \frac{\Gamma(1-(n\alpha)/(4H))}{\Gamma(1-(n\alpha)/2)} x^{(n\alpha)/(4H)}. \tag{4.5}$$

The Lévy-stability property is *not* preserved under the subordination operation, as soon as $H \neq \frac{1}{2}$.

Remark 4. This process should *not* be confused with the linear fractional stable motion (FSM) defined by Taqqu and Wolpert [35, 38] as

$$\overline{\mathfrak{L}}_H^{\alpha}(t) := \int_{\mathbb{R}} ((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha})\,\mathrm{d}\overline{B}^{\alpha}(s) \qquad H \neq 1/\alpha \tag{4.6}$$

extending the moving average Wiener representation of FBM (2.1) to Lévy driving integrators, and with adequate integrands. This process is sαs and H-SS and has not much in common with $\overline{B}_H^{\alpha}(t)$. Note, however, that $\overline{\mathfrak{L}}_H^2(t) \equiv \overline{B}_H(t)$, whereas $\overline{B}_{1/2}^{\alpha}(t) \equiv \overline{B}^{\alpha}(t)$: the FSM reduces to FBM as $\alpha = 2$, whereas FLM reduces to standard LM as $H = \frac{1}{2}$.

For the FSM, the characteristic function of the one-dimensional marginal may be shown to be $\mathrm{e}^{-t^{\alpha H}|\lambda|^{\alpha}}$, if $t > 0$, hence stable (see, e.g., proposition 7.4.3 of [35]). It follows that $\boldsymbol{P}(\overline{\mathfrak{L}}_H^{\alpha}(t) > x) \propto t^{\alpha H} x^{-\alpha}$ as $x \uparrow +\infty$, and that $\boldsymbol{D}(\overline{\mathfrak{L}}_H^{\alpha}(t)) \underset{t\uparrow+\infty}{\propto} t^H$, which is similar to the behaviour of FBM. If $1/\alpha < H < 1$, and $\alpha \in (1,2)$, this process may be shown to present long-range dependence, in some extended sense [35]. Clearly, in this approach, the stable character of the resulting process has been favoured and this model is the natural extension of the FBM *in this respect*. However, forcing the stable character has one main drawback: it cannot represent jump processes with finite variances which seems to be a practical limitation in some applications (including finance). Equation (4.1) goes in this different direction: finite-variance processes are allowed for by a judicious choice of the parameters (α, H) (see below); however, the stability property had to be abandoned.

Let us now exhibit some additional properties of $\overline{B}_H^{\alpha}(t)$.

If $F_H^{\alpha}(t,x) := \boldsymbol{P}(\overline{B}_H^{\alpha}(t) \leqslant x)$ is the PDF of $\overline{B}_H^{\alpha}(t)$, and $\overline{F}_H^{\alpha}(t,x) := 1 - F_H^{\alpha}(t,x)$ the complementary PDF, we have the heavy-tail behaviour

$$\overline{F}_H^{\alpha}(t,x) \underset{x\uparrow+\infty}{\propto} t x^{-\alpha/(2H)}. \tag{4.7}$$

This conclusion may be derived in the following simple way.

We first note from (3.18) that $\boldsymbol{E}\mathrm{e}^{-p\overline{S}^{\alpha/2}(t)} = \mathrm{e}^{-t(2p)^{\alpha/2}} \underset{p\uparrow 0^+}{\sim} 1 - t(2p)^{\alpha/2}$. As a result, we have the hyperbolic tail behaviours $\boldsymbol{P}(\overline{S}^{\alpha/2}(t) > s) \underset{s\uparrow+\infty}{\propto} ts^{-\alpha/2}$ and $\boldsymbol{P}(\overline{S}^{\alpha/2}(t)^{2H} > s) \underset{s\uparrow+\infty}{\propto} ts^{-\alpha/(4H)}$, raising the variable $\overline{S}^{\alpha/2}(t)$ to the power $2H$. Hence, $\boldsymbol{E}\mathrm{e}^{-p\overline{S}^{\alpha/2}(t)^{2H}} \underset{p\uparrow 0^+}{\propto} 1 - t(2p)^{\alpha/(4H)}$.

Now, from (4.3), $\overline{\Phi}_H^{\alpha}(t,\lambda) = \boldsymbol{E}\mathrm{e}^{-p\overline{S}^{\alpha/2}(t)^{2H}}$ $(p = \lambda^2/2)$, so that, for some appropriate constant $K > 0$,

$$\overline{\Phi}_H^{\alpha}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}_H^{\alpha}(t)} \underset{\lambda\uparrow 0}{\sim} 1 - Kt|\lambda|^{\alpha/(2H)} \qquad \alpha \in (0,2) \quad H \in (0,1). \tag{4.8}$$

The control of the tail behaviour (4.7) follows.

It should be noted from (4.7) that the larger $(2H)/\alpha$ is, the heavier the tails of the process $\overline{B}_H^\alpha(t)$ are. If $(2H)/\alpha < \frac{1}{2}$, the process $\overline{B}_H^\alpha(t)$ has finite variance. If $\frac{1}{2} < (2H)/\alpha < 1$, the process $\overline{B}_H^\alpha(t)$ has finite absolute mean $\boldsymbol{E}|\overline{B}_H^\alpha(t)| < +\infty$ but is with infinite variance. If $(2H)/\alpha > 1$, the absolute mean itself diverges. These tail properties are in sharp contrast to those of the FSM (4.6) which are *always* with infinite variance as $\alpha \in (0, 2)$.

Thus, again, the definition of FLM (equation (4.1)) yields SS with SIs jump processes which are not stable. These processes are heavy tailed but not necessarily with infinite variances: the tail index $\alpha/(2H)$ may vary on the whole positive real line and is *not* limited to the interval $(0, 2)$. Allowing for such a richer tail behaviour led us to abandon the stability condition in the definition of a FLM, which on the contrary is an essential feature of the FSM.

Let us now derive information relating to the asymptotic dispersion of $\overline{B}_H^\alpha(t)$.

The median function of $\overline{B}_H^\alpha(t)$ clearly is the null function for symmetry reasons, whereas the asymptotic form of the (quantile) dispersion of $\overline{B}_H^\alpha(t)$, $\boldsymbol{D}(\overline{B}_H^\alpha(t)) := d_H^\alpha(t)$, is

$$\boldsymbol{D}(\overline{B}_H^\alpha(t)) \underset{t\uparrow+\infty}{\propto} t^{(2H)/\alpha} \tag{4.9}$$

from the tail behaviour (4.7). Note that the equation $H = \alpha/4$ separates the subdiffusive region ($H < \alpha/4$) from the superdiffusive one ($H > \alpha/4$): we shall call this the diffusive region of $\overline{B}_H^\alpha(t)$. Thus $\overline{B}_{\alpha/4}^\alpha(t)$, $\alpha \in (0, 2)$ is a class of FLM with diffusive properties similar to that of standard BM although, of course, of a completely different statistical nature. This can be achieved at the only condition that $H < \frac{1}{2}$ (the underlying FBM has to be subdiffusive). For these processes, the antipersistence character which forces the underlying FBM to meander exactly compensates for the superdiffusivity property arising from the heavy-tailed character of the resulting process. Note that a special role in this family is played by $\overline{B}_{1/4}^1(t)$ obtained for $\alpha = 1$, which may be called fractional Cauchy motion.

Let us finally consider the structure of correlations, which is poorly understood at present. We shall limit ourselves to the case $0 < (2H)/\alpha < 1$ and distinguish between $0 < (2H)/\alpha < \frac{1}{2}$ and $\frac{1}{2} < (2H)/\alpha < 1$.

- $0 < (2H)/\alpha < \frac{1}{2}$. This is the finite-variance case. It follows from lemma 7.2.1 of [35] that, in this case, $\overline{B}_H^\alpha(t)$ is a finite variance $(2H)/\alpha$-SS with SIs process. As a result, it admits the covariance function given in (2.3) where $2H/\alpha$ should be substituted to H. Concerning the increment process $B_H^\alpha(t)$, its covariance function is given by (2.4) where $2H/\alpha$ is again substituted for H. As $2H/\alpha < \frac{1}{2}$, the covariance function has power-law decay but the spectral density does not diverge at $f = 0$ (see (2.6)): we have no long-range dependence there, simply negative dependence.
- $1 > (2H)/\alpha > \frac{1}{2}$. In this case, the temporal dependence structure cannot be measured in terms of the covariance function (it is ill defined due the presence of too large jumps). This is also the case in the FSM model for which the notions of covariation or codifference have to be used instead [35]. However, these notions are adapted to stable processes and the FLM is (again) not stable: these notions are inappropriate in our case. We propose to substitute this notion with that of the quantile covariance function (see below for a definition), say $\boldsymbol{Q}$, which is *conjectured* to behave according to

$$\boldsymbol{Q}[B_H^\alpha(s), B_H^\alpha(s+t)] \underset{t\uparrow+\infty}{\sim} t^{2((2H)/\alpha-1)} \tag{4.10}$$

  in the parametric domain $\frac{1}{2} < (2H)/\alpha < 1$. The quantile spectral density, defined as the Fourier transform of the quantile covariance function should therefore behave as $1/f^{4H/\alpha-1}$ in the vicinity of the frequency $f = 0$, showing the divergence syndrome in this region.

We now define the quantile correlation notion used in (4.10) which is believed to play a central role here. The quantile covariance $\boldsymbol{Q}(X, Y)$ of two dependent variables X and Y, with respective medians m_X and m_Y, and respective dispersions d_X and d_Y may be defined in the following way. Consider the two-dimensional extension of (3.7):

$$\boldsymbol{P}(|X - m_X| > x, |Y - m_Y| > y) = \tfrac{1}{2}. \tag{4.11}$$

This equation generates a one-dimensional curve, say C, in the $(x \geqslant 0, y \geqslant 0)$ quadrant; this curve intersects the axis $y = 0$ and $x = 0$ at points with coordinates $(d_X, 0)$ and $(0, d_Y)$. Next, define the curve C' in the following implicit way:

$$\boldsymbol{P}(|X - m_X| > x)\boldsymbol{P}(|Y - m_Y| > y) = \tfrac{1}{2}. \tag{4.12}$$

The curve C' also intersects the axis $y = 0$ and $x = 0$ at points with coordinates $(d_X, 0)$ and $(0, d_Y)$. Now, if (X, Y) is an independent vector, the curves C and C' coincide; on the other hand, in the case of total dependence where Y is a deterministic function of X, (4.11) reduces to $\boldsymbol{P}(|X - m_X| > \max(x, y)) = \frac{1}{2}$ so that curve C identifies with the segments $(x = d_X, 0 \leqslant y \leqslant d_Y) \cup (0 \leqslant x \leqslant d_X, y = d_Y)$. This suggests that the quantile covariance $\boldsymbol{Q}(X, Y)$ of two variables X and Y may be defined to be the area enclosed between C and C'. Note that this 'correlation' notion may also be used for jointly stable vectors themselves.

5. Related stationary processes

5.1. Generalized Ornstein–Uhlenbeck processes

So far, we have been concerned with 'free' motions, either Brownian or Lévy. It is interesting to consider such motions in the linear force (Langevin) context. We shall formulate this problem using the so-called Lamperti transform, inspired by group theory. This approach emphasizes the fact that self-similarity and stationarity are closely related: an exponential time transform translates scale invariance into shift invariance of the stationary process.

As is well known from [19], the process

$$U_{1/2}(t) := \sigma \mathrm{e}^{-at/2} \cdot \overline{B}_{1/2}(\mathrm{e}^{at}) \qquad a > 0 \quad \sigma > 0 \tag{5.1}$$

identifies with the standard Ornstein–Uhlenbeck process [41,42], which describes the motion of a Brownian particle in the harmonic potential $\frac{1}{2}ax^2$. Alternatively, this process may be viewed as the solution of a stochastic differential equation with a linear drift, driven with additive Brownian noise. It is stationary as a result of the classical property $\boldsymbol{E}[\overline{B}_{1/2}(s)\overline{B}_{1/2}(t)] = \inf(s, t)$ of BM, so that

$$\boldsymbol{E}[U_{1/2}(s)U_{1/2}(s+t)] = \sigma^2 \mathrm{e}^{-a|t|} \tag{5.2}$$

showing now exponential decay.

It is therefore possible to define in a similar way the fractional Gauss–Ornstein–Uhlenbeck process as

$$U_H(t) := \sigma \mathrm{e}^{-aHt} \cdot \overline{B}_H(\mathrm{e}^{at}) \qquad da > 0 \quad \sigma > 0 \tag{5.3}$$

whose covariance function can be shown to behave asymptotically as

$$\boldsymbol{E}[U_H(s)U_H(s+t)] \sim \sigma^2 \mathrm{e}^{-2aH|t|} \qquad \text{as} \quad |t| \uparrow +\infty \tag{5.4}$$

as a result of the expression (2.3) for the covariance function of the FBM.

Both standard Ornstein–Uhlenbeck and fractional Gauss–Ornstein–Uhlenbeck processes share the same Gaussian invariant DF; however, they have different autocovariance function and may be discriminated in this way.

Following the same guiding lines, the Ornstein–Uhlenbeck–Lévy process

$$U^{\alpha}(t) := \sigma e^{-at/\alpha} \cdot \overline{B}^{\alpha}(e^{at}) \qquad a > 0 \quad \sigma > 0 \tag{5.5}$$

may been considered, leading to non-Gibbsian (or non-Boltzmann) stationary solutions. For these processes, from (3.1),

$$\boldsymbol{E} e^{i\lambda U^{\alpha}(t)} = \overline{\Phi}^{\alpha}(e^{at}, \lambda\sigma e^{-at/\alpha}) = \exp -e^{at}|\lambda\sigma e^{-at/\alpha}|^{\alpha} = \exp -|\lambda\sigma|^{\alpha} \tag{5.6}$$

independent of time, with a non-Gaussian invariant DF, but of Lévy type, with infinite variance (see [14] for a similar account but in a different language; see also [39, 40]).

We finally define the fractional Ornstein–Uhlenbeck–Lévy process as

$$U_H^{\alpha}(t) := \sigma e^{-2aHt/\alpha} \cdot \overline{B}_H^{\alpha}(e^{at}) \qquad a > 0 \quad \sigma > 0. \tag{5.7}$$

For this kind of process, from (4.4) and property (3.20)

$$\overline{\Phi}_H^{\alpha}(t, \lambda) := \boldsymbol{E} e^{i\lambda \overline{B}_H^{\alpha}(t)} = \int_0^{+\infty} e^{-s^{2H}\lambda^2/2} g^{\alpha}(t, s)\, ds = \overline{\Phi}_H^{\alpha}(1, t^{2H/\alpha}\lambda). \tag{5.8}$$

As a result,

$$\boldsymbol{E} e^{i\lambda U_H^{\alpha}(t)} = \overline{\Phi}_H^{\alpha}(e^{at}, \lambda\sigma e^{-2aHt/\alpha}) = \overline{\Phi}_H^{\alpha}(1, \sigma\lambda) = \int_0^{+\infty} e^{-s^{2H}(\sigma\lambda)^2/2} g^{\alpha}(s)\, ds \tag{5.9}$$

characterizes its invariant DF. We shall call this distribution the symmetric *fractional* Lévy DF. Note that it is *not* a Lévy-stable distribution, so that this property discriminates the process (5.7) from the standard Ornstein–Uhlenbeck–Lévy process (5.5). The symmetric *fractional* Lévy DF exhibits power-law tails with index $\alpha/(2H) > 0$; it has the characteristic function given by (5.9), which, from (4.4), may be written as $\psi_{\alpha,H}((\sigma\lambda)^2/2)$ in terms of the function $\psi_{\alpha,H}$.

5.2. Multiplicative FLM processes

In this section, we would like to briefly introduce an interesting class of processes where FLM may find application; we adopt financial language for historical reasons but it certainly is not limitative: there will be meaningful applications in other areas, such as physics and population biology—wherever some highly fluctuating process (such as the price process) accumulates fluctuations and is bound to remain positive for some physical reasons. In such models it is postulated that there is a regulation mechanism (an arbitrage) which sooner or later forces the physical phenomenon to increase when it is close to becoming extinct (i.e. close to zero).

Let, a and $\sigma > 0$ be some constants. Define the following multiplicative FLM:

$$\overline{P}_H^{\alpha}(t) := \exp(at + \sigma \overline{B}_H^{\alpha}(t)) > 0. \tag{5.10}$$

If $H = \frac{1}{2}$, $\alpha = 2$, $\overline{B}_H^{\alpha}(t)$ reduces to standard BM, so that the model $\overline{P}_{1/2}(t)$ identifies with the Black–Scholes model for a cumulative financial price process at time t (the so-called geometric BM) [2]. In this model, the logarithm of the cumulative price of a risky asset $\log \overline{P}_{1/2}(t)$ is modelled to be the result of interactions caused by a large number of individual traders: i.e., it is understood as a cumulative BM process, and σ is known as the volatility of the asset.

Therefore, model (5.10) is a generalization of the Black–Scholes model for cumulative financial prices in situations where standard BM is replaced by FLM.

The incremental process $P_H^{\alpha}(t) := \overline{P}_H^{\alpha}(t+1) - \overline{P}_H^{\alpha}(t)$ over (say) a unit period of time indicates the daily variation of price. The relative variation of price (a rate) at time t, say $R_H^{\alpha}(t)$, is therefore the strongly asymmetric process

$$R_H^{\alpha}(t) := \frac{P_H^{\alpha}(t)}{\overline{P}_H^{\alpha}(t)} = e^{a + \sigma B_H^{\alpha}(t)} - 1 > -1. \tag{5.11}$$

In contrast, some symmetry is found again for the daily log returns

$$L_H^\alpha(t) := \log \frac{\overline{P}_H^\alpha(t+1)}{\overline{P}_H^\alpha(t)} \tag{5.12}$$

of the price process, which is found to be

$$L_H^\alpha(t) = \log(1 + R_H^\alpha(t)) = a + \sigma B_H^\alpha(t) \tag{5.13}$$

in terms of the increments of FLM, after a location-scale transform. In such models, both relative variation and log-return processes are stationary and $L_H^\alpha(t) \sim R_H^\alpha(t)$ at times where $B_H^\alpha(t)$ is small. One of the main advantages in the log differencing of financial data is that this operation makes them comparable as they become independent of the monetary unit and the hypothesis of stationarity of the log returns is currently accepted as a satisfactory working hypothesis—at least with a judicious choice of the period.

Replacing standard BM in the Black–Scholes model by FLM, as in (5.10), amounts to refuting the two simplifying hypotheses: that the series of log returns are light tailed (here normal) and that they are without complicated dependence structure in time; the observations show convincingly enough that both hypotheses have to be rejected: a glance at any series of daily log returns indeed shows that there are some values much larger than the others, supporting the evidence of 'heavy tailedness' [27,28]. Besides, the infinite variance hypothesis has gained only marginal popularity: some reports on data with finite variance and infinite third or fourth moment exist. This supports the pertinence of FLM which may exhibit either finite or infinite variance depending on its tail index $\alpha/(2H)$.

However, (5.10) is only one of the possible models for these data; others exist, such as ARCH models [7] which extend the Black–Scholes model in a different direction: assuming the volatility to be a complex function of the log returns from the past.

We shall not enter into more detail here, because the central problem is statistical and consists of the difficult problem of estimating both H and α from real data. However, we shall add a final remark: were (5.11) to be a 'good' model for relative variation of price in financial series, then the tails of the process $R_H^\alpha(t)$ would be 'extremely heavy' at all times, in the sense that the tail probability of $R_H^\alpha(t)$ would tend to zero *slower* than any power-law $r^{-\gamma}$, for any $\gamma > 0$: such distributions are said to be heavy tailed with index zero. This arises from the fact that exponentiating a random variable, as in (5.11), fattens its tails in a drastic way.

Indeed, from $R_H^\alpha(t) \overset{d}{=} R_H^\alpha(1) = \mathrm{e}^{a+\sigma B_H^\alpha(1)} - 1$ we get, if $\alpha \neq 2$, from (4.7),

$$\boldsymbol{P}(R_H^\alpha(t) > r) = \boldsymbol{P}(B_H^\alpha(1) > 1/\sigma(\log(1+r) - a)) \underset{r\uparrow+\infty}{\sim} 1/(\sigma \log r)^{\alpha/(2H)} \tag{5.14}$$

so that the tail distribution of $R_H^\alpha(t)$ is heavy tailed with index zero, in the sense that, for any $\gamma > 0$,

$$\frac{\boldsymbol{P}(R_H^\alpha(t) > r)}{r^{-\gamma}} \underset{r\uparrow+\infty}{\rightarrow} +\infty. \tag{5.15}$$

Very large positive fluctuations allow the process to reset when it gets close to extinction.

6. FT-BMs

Apart from FLM, all the processes that have been considered in sections 2–4 (that is BM, FBM, LM, FSM) have SI and share the property of stability under addition (either Gaussian or α-stable); they are also all SS, as a consequence of the stability property. They are thus SS-SI and stable.

Thus, FLM is a natural example of a process which is *not* stable; this property is lost while subordinating FBM to a SS-SI Lévy subordinator. However, FLM remains SS-SI. As a result, all processes discussed so far (that is BM, FBM, LM, FLM, FSM) are all SS with SIs (SS-SI).

In this section, we introduce a one-parameter δ-family of 'fractional' processes for which a weaker notion of self-similarity seems to hold, i.e. self-similarity of the unidimensional distributions: this is fractal time BM (FT-BM). Such a process is obtained as a weak limit of fractal time random walk models, with $\delta \in (0, 1)$ the tail exponent of the waiting times, and makes use of some simple notions borrowed from renewal process theory [8].

FT-BM identifies with BM subordinated to a fractal inverse Lévy subordinator, which, in sharp contrast to the SS-SI Lévy subordinator, is *not* a SS-SI process.

This construction extends to FBM, LM and FLM: we therefore introduce and study FT-FBM, FT-LM and FT-FLM which are the fractal time extensions of FBM, LM and FLM.

In sharp contrast to BM, FBM, LM, FLM, these FT-extensions are neither strongly SS, nor with SIs. And, just as FLM, they are not stable.

6.1. Recurrent renewal processes: continuous time random walks

Let a cumulative stochastic process $\overline{X}(t)$ be defined in distribution by

$$\overline{X}(t) \stackrel{d}{=} 0 \cdot \mathbf{1}(T > t) + (X(T) + \overline{X}(t - T)) \cdot \mathbf{1}(T \leqslant t) \tag{6.1}$$

where $T > 0$ is a positive random variable known as the first renewal time of $\overline{X}(t)$. Time T is assumed to be a random variable with a proper (i.e. with total mass unity) DF, say $f_T(\cdot)$. Such processes are called pure recurrent renewal processes.

Let us briefly comment on this formula. At time T, $\overline{X}(t)$ undergoes a first (random) jump with amplitude $X(T)$, possibly dependent on the occurrence time T of this jump. Fix time t at which $\overline{X}(t)$ is to be evaluated. If the realization of time T exceeds the time t of interest, the process $\overline{X}(t)$ is in the initial state (zero here). If $T \leqslant t$, the value of $\overline{X}(t)$ is the independent sum of the first jump of amplitude $X(s)$ plus a statistical copy of the process $\overline{X}(\cdot)$ in the remaining time $t - s$, conditionally to the event $T = s$. This is a reasonable way to see a renewal process: it generalizes the familiar compound Poisson process family in that the inter-arrival time distributions between spikes is an *iid* sequence but not necessarily exponentially distributed.

Let us now translate the definition (6.1) in terms of the evolution of the characteristic function of $\overline{X}(t)$. Let

$$\overline{\Phi}(t, \lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{X}(t)} \qquad \text{and} \qquad \phi_X(s, \lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda X(s)} \tag{6.2}$$

respectively stand for the characteristic functions of the cumulative process $\overline{X}(t)$ and of a local increment $X(s)$ which occurred at time $s \leqslant t$. Then

$$\overline{\Phi}(t, \lambda) = \boldsymbol{P}(T > t) + \int_0^t \overline{\Phi}(t - s, \lambda)\phi_X(s, \lambda) f_T(s)\,\mathrm{d}s. \tag{6.3}$$

We shall now make an additional simplifying hypothesis.

Assume that the local amplitudes are independent of their occurrence time, so that $\phi_X(s, \lambda) = \phi_X(\lambda)$: the characteristic function of the conditional increment $X(s)$ is independent of the particular realization s of the occurrence time T. Then (6.3) reduces to

$$\overline{\Phi}(t, \lambda) = \boldsymbol{P}(T > t) + \phi_X(\lambda) \int_0^t \overline{\Phi}(t - s, \lambda) f_T(s)\,\mathrm{d}s. \tag{6.4}$$

This is the integral (convolution) equation that $\overline{\Phi}(t,\lambda)$ now satisfies. Introducing the Laplace transforms

$$\overline{\Phi}(p,\lambda) := \int_0^{+\infty} \mathrm{e}^{-pt}\overline{\Phi}(t,\lambda)\,\mathrm{d}t \qquad \text{and} \qquad \phi_T(p) := \int_0^{+\infty} \mathrm{e}^{-ps} f_T(s)\,\mathrm{d}s \tag{6.5}$$

of $\overline{\Phi}(\cdot,\lambda)$ and $f_T(\cdot)$, respectively, yields

$$\overline{\Phi}(p,\lambda) = \frac{1-\phi_T(p)}{p(1-\phi_T(p)\phi_X(\lambda))} = \frac{1}{p(1+\frac{\phi_T(p)}{1-\phi_T(p)}(1-\phi_X(\lambda)))} \tag{6.6}$$

provided that $\phi_T(p)\phi_X(\lambda) < 1$.

Processes whose Laplace transform of the characteristic function satisfies this equation are known in the literature of physics as continuous time random walks (CTRW). If the inter-arrival time T is heavy tailed in such a way that it has infinite mean value, the process is referred to in the literature as fractal time random walk (FTRW) [15, 17, 31, 43].

These CTRW model some physical phenomenon that is to be described in the following way: events of random *iid* magnitudes, say $(X_m)_{m\geqslant 1}$, occur at random times $\overline{T}_n$, $n \geqslant 1$, the inter-arrival times of which, say $T_m := \overline{T}_m - \overline{T}_{m-1}$, $m \geqslant 1$, form an *iid* sequence, with $T_m \stackrel{d}{=} T$, $m \geqslant 1$. The process $\overline{X}(t)$ cumulates the individual magnitudes which occurred before time t and $\overline{X}(t) = \sum_{m=1}^{\overline{N}(t)} X_m$, with $\overline{N}(t)$ the random number of events which occurred at time t.

CTRWs are also called *pure* renewal processes in the literature of probability theory (the adjective pure is relative to the hypothesis which has been made that the origin of time is an instant at which some event occurred; if this not the case, the adjective *delayed* is currently employed and the first event occurs at time $\overline{T}_0 := T_0 > 0$, independent of $(T_m)_{m\geqslant 1}$ but not necessarily with the same distribution). If, in addition, $\int_0^{+\infty} f_T(s)\,\mathrm{d}s = 1$ (T is 'proper') such renewal processes are said to be recurrent; this has to be opposed to transient renewal processes for which $\int_0^{+\infty} f_T(t)\,\mathrm{d}t < 1$, corresponding to 'defective' T, allowing for a finite probability that the first event never occurs, i.e. occurs at time $t = +\infty$. We shall avoid transient processes in what follows and limit ourselves to recurrent ones. However, among recurrent processes, we shall distinguish between positive recurrent processes for which the average renewal time $\boldsymbol{E}T := \theta < +\infty$ and null recurrent for which $\boldsymbol{E}T = +\infty$. If $\boldsymbol{E}T = +\infty$, we shall limit ourselves in this article to situations where this occurs as a result of 'heavy-tailedness' of the inter-arrival time: $\overline{F}_T(t) \sim c_\delta t^{-\delta}$, as $t \uparrow +\infty$, with $\delta \in (0,1)$. Here, $c_\delta > 0$ is a scale factor for T. In other words $c_\delta = t_0^\delta$ for some $t_0 > 0$ fixing the timescale itself. Note that c_δ has the dimension of time raised to the power δ.

Remark 5. This generalization has a drastic impact on the increment process

$$X(t,t+s) := \overline{X}(t+s) - \overline{X}(t) \qquad \text{with} \quad s,t > 0.$$

This process may be generated in the following self-coherent way, using the forward recurrence time notion, say $F(t)$, which is the random time separating current time t from the next impulsion:

$$X(t,t+s) \stackrel{d}{=} 0\cdot \mathbf{1}(F(t) > s) + (X + X(t+F(t),t+s))\cdot \mathbf{1}(F(t) \leqslant s).$$

From this expression, one immediately realizes that, except for very particular cases (T exponentially distributed), the distribution of $X(t,t+s)$ will depend on both s and t, and not on s only: the increment process is not stationary in general (except maybe in some asymptotic sense). Note also that the forward recurrence time $F(t)$ may itself be generated through

$$F(t) \stackrel{d}{=} (T-t)\cdot \mathbf{1}(T > t) + F(t-T)\cdot \mathbf{1}(T \leqslant t).$$

6.2. Positive recurrence in the random walk limit: BM

CTRWs are jump processes, in the sense that the sample paths are discontinuous. Let us now consider a limiting situation where some continuity of the sample paths will be found.

Suppose the jump's amplitude X has the very peculiar distribution

$$\frac{1}{2} \cdot \delta(x - \varepsilon) + \frac{1}{2} \cdot \delta(x + \varepsilon)$$

with $\epsilon > 0$ and $\delta(x-\varepsilon)$ the Dirac mass at $x = \epsilon$ (the random walk model). Suppose, in addition, that the renewal time T has a density whose first moment is finite, say $\boldsymbol{E}(T) := \theta < +\infty$ (the property of positive recurrence). The constant θ is thus the expected time between two consecutive increments of $\overline{X}(\cdot)$. Then the Laplace transform of $f_T(\cdot)$ admits the representation

$$\phi_T(p) = 1 - \theta p + \mathrm{o}(\theta p) \tag{6.7}$$

where $\lim \mathrm{o}(x)/x = 0$, as $x \uparrow 0^+$. In the limit $\theta \uparrow 0^+$ and $\epsilon \uparrow 0^+$ (small jumps occur at infinite rate), while $D_{1/2} := \epsilon^2/\theta > 0$ (the diffusion constant) is held fixed,

$$\overline{\Phi}(p, \lambda) = \frac{1}{p + D_{1/2}\lambda^2/2} \tag{6.8}$$

so that $\overline{\Phi}(t, \lambda) = \exp -D_{1/2}t\lambda^2/2$.

This is the characteristic function of a centred Gaussian density with variance $D_{1/2}t$. The process $\overline{X}(t)$ therefore boils down to BM $\overline{B}_{1/2}(t)$ with diffusion constant $D_{1/2}$, whatever the particular form of f_T. The limit process is thus standard BM, which is SS-SI and stable.

6.3. Null recurrence in the random walk limit: FT-BM

Let us now consider a different limiting situation, which is more degenerate and more interesting.

Assume now that, with $\delta \in (0, 1)$,

$$\phi_T(p) = 1 - c_\delta p^\delta + \mathrm{o}(c_\delta p^\delta). \tag{6.9}$$

This form of $\phi_T(p)$ indicates that the renewal time T now only possesses fractional moments of order less than $\delta < 1$: in other words $\boldsymbol{P}(T > t) \underset{t\uparrow+\infty}{\sim} c_\delta t^{-\delta}$ and the renewal time is heavy-tailed (with, in particular, $\boldsymbol{E}(T) = +\infty$); this is the null recurrence hypothesis of the underlying renewal process. This is a *rare event* hypothesis in the primary sense and such renewal processes identify with FTRWs.

In the limit $c_\delta \uparrow 0^+$ and $\epsilon \uparrow 0^+$, while $D_\delta := \epsilon^2/c_\delta > 0$ (the generalized diffusion constant) is held fixed, we easily get from (6.6), (6.9) the limit form

$${}^\delta\overline{\Phi}(p, \lambda) = \frac{1}{p(1 + D_\delta p^{-\delta}\lambda^2/2)}. \tag{6.10}$$

Finally, the limit process, ${}^\delta\overline{B}(t)$, has continuous sample paths (as a random walk limit); however, it is *not* stable. Indeed, from (6.10), we get its characteristic function

$${}^\delta\overline{\Phi}(t, \lambda) = \phi_\delta(D_\delta t^\delta \lambda^2/2) \qquad \text{with} \quad \phi_\delta(x) = \sum_{n\geqslant 0} \frac{1}{\Gamma(1 + n\delta)}(-x)^n \tag{6.11}$$

which is neither Gaussian nor α-stable. In (6.11), the function $\phi_\delta(x)$ is identified with the (entire) Mittag–Leffler function, which reduces to $\exp -x$ as $\delta = 1$ (the previous case); as a result, from (6.11), this process has moments of arbitrary order and, in particular, a finite variance and

$$\boldsymbol{\sigma}[{}^\delta\overline{B}(t)] \propto t^{\delta/2}. \tag{6.12}$$

As $\delta < 1$, ${}^{\delta}\overline{B}(t)$ is always subdiffusive. We shall call this process the FT-BM. As conventional wisdom suggests, FT-BM is less diffusive as it is in 'universal' clock time t.

We now show the following important property.

FTBM ${}^{\delta}\overline{B}(t)$ can be interpreted as BM $\overline{B}_{1/2}(s)$ in (independent) fractal inverse-Lévy-stable time $s = {}^{\delta}\overline{I}(t)$: in other words ${}^{\delta}\overline{B}(t)$ is again a subordinate of BM and

$$ {}^{\delta}\overline{B}(t) = \overline{B}_{1/2}({}^{\delta}\overline{I}(t)) \tag{6.13} $$

but with an inverse-Lévy-stable process as subordinator.

6.3.1. The inverse-Lévy-stable subordinator. To see this, let us first construct the subordinating process ${}^{\delta}\overline{I}(t)$. Consider a cumulative increasing stochastic process $\overline{I}(t)$ defined in distribution as in (6.1):

$$ \overline{I}(t) \overset{d}{=} 0 \cdot \mathbf{1}(T > t) + (I + \overline{I}(t - T)) \cdot \mathbf{1}(T \leqslant t). \tag{6.14} $$

Suppose now that the amplitude I is *positive* with the *degenerate* distribution $\delta(x - \varepsilon^2)$, with $\epsilon > 0$ and $\delta(x - \varepsilon^2)$ the Dirac mass at $x = \epsilon^2$; we still assume heavy-tailedness of the inter-arrival time T, in the preceding sense: $\boldsymbol{P}(T > t) \underset{t \uparrow +\infty}{\sim} c_\delta t^{-\delta}$. Proceeding as previously, in the limit $c_\delta \uparrow 0^+$ and $\epsilon \uparrow 0^+$, while $D_\delta := \epsilon^2 / c_\delta > 0$, we get for this limit a non-decreasing process: from (6.6),

$$ {}^{\delta}\overline{\Phi}(p, \lambda) = \frac{1}{p(1 + D_\delta p^{-\delta}\lambda)}. \tag{6.15} $$

We shall denote such a limit subordinator ${}^{\delta}\overline{I}(t)$, underlining its dependence on δ. It is the inverse-Lévy subordinator in the sense that, with $\overline{S}^{\delta}(\cdot)$ a Lévy subordinator,

$$ {}^{\delta}\overline{I}(t) := \inf(s > 0 : \overline{S}^{\delta}(s) > t) \qquad \text{and} \qquad \boldsymbol{E}\mathrm{e}^{-p\overline{S}^{\delta}(s)} = \exp -D_\delta s p^{\delta}. $$

In addition, with $t > 0$,

$$ {}^{\delta}\overline{I}(t) \overset{d}{=} D_\delta \cdot t^{\delta} \cdot {}^{\delta} I \tag{6.16} $$

where ${}^{\delta}I$ is an inverse Lévy variable obtained while raising a Lévy-stable variable S_δ to the power $-\delta$, which means that

$$ {}^{\delta}I \overset{d}{=} S_\delta^{-\delta} \qquad \text{with} \qquad \boldsymbol{E}\mathrm{e}^{-pS_\delta} = \exp -p^{\delta} \qquad p \geqslant 0. \tag{6.17} $$

Indeed (see [8], vol 2, XIII. 8, (8.4), p 453), we have the Mittag–Leffler formula

$$ \boldsymbol{E}\mathrm{e}^{-\lambda \cdot {}^{\delta}\overline{I}(t)} = \sum_{n \geqslant 0} \frac{(-\lambda D_\delta)^n}{\Gamma(1 + n\delta)} t^{n\delta} = \phi_\delta(D_\delta t^{\delta} \lambda) \tag{6.18} $$

and

$$ \int_0^{+\infty} \mathrm{e}^{-pt} \phi_\delta(D_\delta t^{\delta}\lambda)\, \mathrm{d}t = \frac{1}{p(1 + D_\delta p^{-\delta}\lambda)} \tag{6.19} $$

which coincides with the expression in (6.15), characteristic of ${}^{\delta}\overline{I}(t)$. From (6.18), it is clear that ${}^{\delta}\overline{I}(t)$ is not a SS-SI process: its characteristic function is not even that of an infinitely divisible process.

6.3.2. FT-BM is a subordinate of BM with inverse-Lévy-stable subordinator. Now, to see the subordination property, with $f_{^\delta\overline{I}(t)}(s)$ the density of ${}^\delta\overline{I}(t)$ at point $s > 0$, we have

$$\boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\overline{B}_{1/2}({}^\delta\overline{I}(t))} = \int_0^{+\infty} \mathrm{e}^{-s\frac{\lambda^2}{2}} f_{^\delta\overline{I}(t)}(s)\,\mathrm{d}s = \phi_\delta(D_\delta t^\delta \lambda^2/2). \tag{6.20}$$

Thus,

$${}^\delta\overline{\Phi}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda.{}^\delta\overline{B}_{1/2}(t)} = \phi_\delta(D_\delta t^\delta \lambda^2/2)$$

which is the characteristic function of ${}^\delta\overline{B}(t)$ (6.11). In addition, from (6.19),

$${}^\delta\overline{\Phi}(p,\lambda) := \int_0^{+\infty} \mathrm{e}^{-pt\,\delta}\overline{\Phi}(t,\lambda)\,\mathrm{d}t = \frac{1}{p(1 + D_\delta p^{-\delta}\lambda^2/2)} \tag{6.21}$$

which is (6.10), characteristic of ${}^\delta\overline{B}(t)$.

As ${}^\delta\overline{I}(t)$ has continuous sample paths, from its construction, ${}^\delta\overline{B}(t)$ has continuous sample path, as a composition of two processes with continuous sample paths.

6.3.3. Some additional remarks. The limit process ${}^\delta\overline{B}(t)$, is *weakly* SS with SS parameter $\delta/2 \in (0, \frac{1}{2})$, which means that for any $t, a > 0$, ${}^\delta\overline{B}(at) \stackrel{d}{=} a^{\delta/2} \cdot{}^\delta\, \overline{B}(t)$. Assuming this were the case, ${}^\delta\overline{\Phi}(at,\lambda) = {}^\delta\overline{\Phi}(t, a^{\delta/2}\lambda)$ so that, taking the Laplace transform, $\frac{1}{a}{}^\delta\overline{\Phi}(p/a,\lambda) = {}^\delta\overline{\Phi}(p, a^{\delta/2}\lambda)$ which the above expression (6.10) verifies. This weak self-similarity (for any unidimensional distribution) should not be confused with the stronger one (for *any* finite-dimensional distributions) defined in (1.2) which does *not* hold for this process. In addition, it may be shown that the increments of ${}^\delta\overline{B}(t)$ are *not* stationary in the strong sense (2.2).

For ${}^\delta\overline{B}(t)$ to be strongly SS in the sense (1.2), would require that the increment process ${}^\delta B(t,s) := {}^\delta\overline{B}(t+s) -{}^\delta\, \overline{B}(t)$ would be itself SS, i.e. that

$${}^\delta B(at, as) \stackrel{d}{=} a^{\delta/2} \cdot{}^\delta\, B(t,s) \qquad \text{for all} \quad a > 0$$

which is not true, chiefly because it is *not* true that the increments of ${}^\delta\overline{B}(t)$ are stationary. Indeed, it is easy to show that a weakly SS process with SIs is strongly SS (although the reciprocal is false in general: the example in remark 1 exhibits a strongly H-SS process with unstationary increments).

Note however, from (6.11), that ${}^\delta\overline{B}(t)$ has moments of arbitrary order and has, in particular, a finite variance: we have here an example of a non-Gaussian finite variance weakly $\delta/2$-SS process with unstationary increments.

We also remark that the 'Ornstein–Uhlenbeck' process ${}^\delta U(t) := \mathrm{e}^{-(\delta t)/2} \cdot{}^\delta\, \overline{B}(\mathrm{e}^t)$ is only *weakly* stationary: that is, for any $t, h > 0$, ${}^\delta U(t) \stackrel{d}{=}{}^\delta\, U(t+h)$; the function $\phi_\delta(D_\delta\lambda^2/2)$ is the characteristic function of its invariant unidimensional DF. This should not be confused with the strict stationarity property which requires that any finite-dimensional distributions are invariant under a shift in time. This *weak*-stationarity property is consistent with the weak SS of ${}^\delta\overline{B}(t)$.

6.4. FT-FBM, FT-LMs and FLMs

The construction of FT-BM and those of LM and FLM discussed in this paper allow us to define various processes of physical interest.

First, the FT-FBM, ${}^\delta\overline{B}_H(t)$, may be defined, with δ and $H \in (0, 1)$, as

$${}^\delta\overline{B}_H(t) := \overline{B}_H({}^\delta\overline{I}(t)) \tag{6.22}$$

i.e. as FBM in independent inverse-Lévy-stable time. It has continuous sample paths.

We shall next define the FT-LM, ${}^{\delta}\overline{B}^{\alpha}(t)$, with $\delta \in (0, 1)$ and $\alpha \in (0, 2)$, as

$$ {}^{\delta}\overline{B}^{\alpha}(t) := \overline{B}^{\alpha}({}^{\delta}\overline{I}(t)) = \overline{B}_{1/2}(\overline{S}^{\alpha/2}({}^{\delta}\overline{I}(t))) \tag{6.23} $$

i.e. as LM in independent inverse-Lévy-stable time ${}^{\delta}\overline{I}(t)$. Here, both subordinators $\overline{S}^{\alpha/2}(\cdot)$ and ${}^{\delta}\overline{I}(\cdot)$ are assumed independent. This process is a jump process.

Finally, we define the FT-FLM, ${}^{\delta}\overline{B}_{H}^{\alpha}(t)$, with δ and $H \in (0, 1)$ and $\alpha \in (0, 2)$, as

$$ {}^{\delta}\overline{B}_{H}^{\alpha}(t) := \overline{B}_{H}^{\alpha}({}^{\delta}\overline{I}(t)) = \overline{B}_{H}(\overline{S}^{\alpha/2}({}^{\delta}\overline{I}(t))) \tag{6.24} $$

i.e. as FLM in independent inverse-Lévy-stable time ${}^{\delta}\overline{I}(t)$.

We now enter into the computational details of their unidimensional distributions.

6.4.1. FT-FBM. First, we note that, with $t > 0$

$$ \overline{B}_{H}(t) \stackrel{d}{=} t^{H} \cdot G \tag{6.25} $$

where G is a standard Gaussian random variable. Thus, from (6.16)–(6.22),

$$ {}^{\delta}\overline{B}_{H}(t) \stackrel{d}{=} [{}^{\delta}\overline{I}(t)]^{H} \cdot G \stackrel{d}{=} D_{\delta}^{H} \cdot t^{\delta H} \cdot [{}^{\delta}I]^{H} \cdot G \tag{6.26} $$

and we need to understand the distribution of the product of a standard Gaussian variable with a Lévy-stable variable S_{δ} raised to the power $-\delta H$, as $[{}^{\delta}I]^{H} \stackrel{d}{=} S_{\delta}^{-\delta H}$.

Now, from Bayes' formula, with ${}^{\delta}\overline{\Phi}_{H}(t, \lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda \cdot {}^{\delta}\overline{B}_{H}(t)}$,

$$ {}^{\delta}\overline{\Phi}_{H}(t, \lambda) = \int_{0}^{+\infty} \mathrm{e}^{-\frac{\lambda^{2}}{2}s^{2H}} f_{{}^{\delta}\overline{I}(t)}(s)\,\mathrm{d}s = \boldsymbol{E}\mathrm{e}^{-[{}^{\delta}\overline{I}(t)]^{2H}\frac{\lambda^{2}}{2}}. \tag{6.27} $$

Thus, from (6.16),

$$ {}^{\delta}\overline{\Phi}_{H}(t, \lambda) = \boldsymbol{E}\mathrm{e}^{-D_{\delta}^{2H}t^{2\delta H}\frac{\lambda^{2}}{2}[{}^{\delta}\overline{I}]^{2H}}. \tag{6.28} $$

After some algebraic manipulations similar to those in ([8], vol 2, XIII. 8, (8.4), p 453), we find the generalized Mittag–Leffler formula

$$ {}^{\delta}\overline{\Phi}_{H}(t, \lambda) = \phi_{\delta,H}\left(D_{\delta}^{2H}t^{2\delta H}\frac{\lambda^{2}}{2}\right) \tag{6.29} $$

with

$$ \phi_{\delta,H}(x) = \sum_{n\geqslant 0} \frac{1}{\Gamma(1+2n\delta H)} \frac{\Gamma(1+2nH)}{\Gamma(1+n)}(-x)^{n} \tag{6.30} $$

being the entire generalized Mittag–Leffler function. Note that, if $H = \frac{1}{2}$, $\phi_{\delta,1/2}(x) = \phi_{\delta}(x)$ and if $\delta = 1$, $\phi_{1,H}(x) = \exp -x$.

If $H = \frac{1}{2}$, formulae (6.29), (6.30) reduce, as required, to (6.11). In addition, from (6.29),

$$ \boldsymbol{\sigma}[{}^{\delta}\overline{B}_{H}(t)] \propto t^{\delta H}. \tag{6.31} $$

If $H < \frac{1}{2}$, the FT-FBM is always subdiffusive, whereas if $H > \frac{1}{2}$, the condition $\delta H < \frac{1}{2}$ ($\delta H > \frac{1}{2}$) yields a subdiffusive (superdiffusive) spreading in time of ${}^{\delta}\overline{B}_{H}(t)$. In any case, the fractal time hypothesis tends to 'slow' the dispersion of FBM.

We now come to the FT-LM and FT-FLM processes. Before we derive an elementary study of some of their statistical properties, let us proceed to some random algebra on the subordinators which appear in (6.23), (6.24).

Set $\alpha_{0} = \alpha/2 \in (0, 1)$, the reduced parameter. In (6.23), (6.24), we need some understanding of the process ${}^{\delta}\overline{S}^{\alpha_{0}}(t) := \overline{S}^{\alpha_{0}}({}^{\delta}\overline{I}(t))$ obtained while composing the two independent subordinators. We have, with $t > 0$,

$$ \overline{S}^{\alpha_{0}}(t) \stackrel{d}{=} t^{1/\alpha_{0}} \cdot S_{\alpha_{0}} \tag{6.32} $$

where $S_{\alpha_0} > 0$ is a Lévy random variable for which $\boldsymbol{E}\mathrm{e}^{-\lambda S_{\alpha_0}} = \mathrm{e}^{-\lambda^{\alpha_0}}$, $\lambda \geqslant 0$. Indeed, $\boldsymbol{E}\mathrm{e}^{-\lambda \overline{S}^{\alpha_0}(t)} = \mathrm{e}^{-t\lambda^{\alpha_0}}$, $\lambda \geqslant 0$, as required, and $\boldsymbol{P}(\overline{S}^{\alpha_0}(t) > x) \underset{x\uparrow+\infty}{\sim} tx^{-\alpha}$, while it is known that $\boldsymbol{P}(\overline{S}^{\alpha_0}(t) \leqslant x) \underset{x\uparrow 0^+}{\sim} \exp -tx^{-\alpha_0/(1-\alpha_0)}$.

Next, concerning ${}^{\delta}\overline{I}(t)$, we saw that

$$ {}^{\delta}\overline{I}(t) \overset{d}{=} D_\delta \cdot t^\delta \cdot [{}^{\delta}I] \tag{6.33} $$

where ${}^{\delta}I \overset{d}{=} S_\delta^{-\delta} > 0$ is an inverse Lévy variable. Thus, we get

$$ {}^{\delta}\overline{S}^{\alpha_0}(t) \overset{d}{=} [{}^{\delta}\overline{I}(t)]^{1/\alpha_0} \cdot S_{\alpha_0} \overset{d}{=} D_\delta^{1/\alpha_0} \cdot t^{\delta/\alpha_0} \cdot [{}^{\delta}I]^{1/\alpha_0} S_{\alpha_0}. \tag{6.34} $$

Thus

$$ {}^{\delta}\overline{S}^{\alpha_0}(t) \overset{d}{=} D_\delta^{1/\alpha_0} \cdot t^{\delta/\alpha_0} \cdot \frac{S_{\alpha_0}}{S_\delta^{\delta/\alpha_0}}. \tag{6.35} $$

In addition, from Bayes' formula and from (6.18),

$$ \boldsymbol{E}\mathrm{e}^{-\lambda\cdot {}^{\delta}\overline{S}^{\alpha_0}(t)} = \boldsymbol{E}\mathrm{e}^{-q\,{}^{\delta}\overline{I}(t)}|_{q=\lambda^{\alpha_0}} = \phi_\delta(D_\delta t^\delta \lambda^{\alpha_0}) \tag{6.36} $$

where ϕ_δ is the Mittag–Leffler function. As a result, from (6.19),

$$ \int_0^{+\infty} \mathrm{e}^{-pt}\boldsymbol{E}\mathrm{e}^{-\lambda\overline{S}^{\alpha/2}({}^{\delta}\overline{I}(t))}\,\mathrm{d}t = \frac{1}{p(1+D_\delta p^{-\delta}\lambda^{\alpha/2})}. \tag{6.37} $$

Remark 6. The Mittag–Leffler function, ϕ_δ, enables one to define a positive Mittag–Leffler random variable, M_δ, as follows:

$$ \boldsymbol{P}(M_\delta > x) = \phi_\delta(x^\delta) \qquad x \geqslant 0. $$

This variable appears in our context in the following way: with $\overline{P}(t)$ a standard Poisson process, let

$$ {}^{\delta}\overline{N}(t) := \overline{P}({}^{\delta}\overline{I}(t)) $$

stand for a Poisson process in inverse Lévy time. Let now

$$ {}^{\delta}\overline{T}^{\alpha_0}(s) := \inf(t > 0 : S^{\alpha_0}({}^{\delta}\overline{N}(t)) > s) $$

stand for the first time at which the increment process

$$ S^{\alpha_0}({}^{\delta}\overline{N}(t)) := \overline{S}^{\alpha_0}({}^{\delta}\overline{N}(t)+1) - \overline{S}^{\alpha_0}({}^{\delta}\overline{N}(t)) $$

exceeds $s > 0$. Physically, this is a 'time between failure' variable.

The events '${}^{\delta}\overline{T}^{\alpha_0}(s) > t$' and '$\max_{s_1\leqslant {}^{\delta}\overline{I}(t)} S^{\alpha_0}(s_1)$' obviously coincide. Thus, upon conditioning this extreme value problem,

$$ \begin{aligned} \boldsymbol{P}\Big(\max_{s_1\leqslant {}^{\delta}\overline{I}(t)} S^{\alpha_0}(\overline{P}(s_1)) \leqslant s\Big) &= \int_0^{+\infty} \mathrm{e}^{-\boldsymbol{P}(S_{\alpha_0}>s)s_2} f_{{}^{\delta}\overline{I}(t)}(s_2)\,\mathrm{d}s_2 \\ &= \phi_\delta(D_\delta t^\delta \lambda)|_{\lambda=\boldsymbol{P}(S_{\alpha_0}>s)} = \boldsymbol{P}({}^{\delta}\overline{T}^{\alpha_0}(s) > t). \end{aligned} $$

As both $(s,t)\uparrow+\infty$, observing that $\boldsymbol{P}(S_{\alpha_0} > s) \sim s^{-\alpha_0}$, we get

$$ \boldsymbol{P}({}^{\delta}\overline{T}^{\alpha_0}(s) > t) \underset{(s,t)\uparrow+\infty}{\sim} \phi_\delta(D_\delta t^\delta s^{-\alpha_0}) = \boldsymbol{P}(M_\delta > D_\delta^{1/\delta} t s^{-\alpha_0/\delta}). $$

Thus, we have shown that

$$ D_\delta^{1/\delta} \cdot s^{-\alpha/(2\delta)} \cdot {}^{\delta}\,\overline{T}^{\alpha/2}(s) \underset{s\uparrow+\infty}{\overset{d}{\to}} M_\delta $$

and the Mittag–Leffler random variable appears as the limit of 'the time between failure' variable ${}^{\delta}\overline{T}^{\alpha/2}(s)$, properly rescaled. From this expression, it is clear that the larger α is, the thinner the tails of $S^{\alpha/2}(\cdot)$ are and the longer one should wait before crossing the failure threshold s; in the same way, the smaller δ is, the 'slower' the process ${}^{\delta}\overline{I}(t)$ is, and the longer this waiting time.

Note that if $\overline{T}^{\alpha_0}(s) := \inf(t > 0 : S^{\alpha_0}(\overline{P}(t)) > s)$ is the time between failure of the sole process $S^{\alpha_0}(\overline{P}(t)) := \overline{S}^{\alpha_0}(\overline{P}(t)+1) - \overline{S}^{\alpha_0}(\overline{P}(t))$, proceeding in the same way yields the convergence result

$$s^{-\alpha/2} \cdot \overline{T}^{\alpha/2}(s) \underset{s\uparrow+\infty}{\overset{d}{\rightarrow}} E$$

with E an exponentially distributed variable: $\boldsymbol{P}(E > t) = \exp -t$.

Let us now investigate the unidimensional distribution of ${}^{\delta}\overline{B}^{\alpha}(t)$ and ${}^{\delta}\overline{B}^{\alpha}_{H}(t)$.

6.4.2. FT-LM. From the obvious property, with $t > 0$,

$$\overline{B}_{1/2}(t) \overset{d}{=} t^{1/2} \cdot G \tag{6.38}$$

and

$${}^{\delta}\overline{S}^{\alpha_0}(t) \overset{d}{=} D_{\delta}^{1/\alpha_0} \cdot t^{\delta/\alpha_0} \cdot \frac{S_{\alpha_0}}{S_{\delta}^{\delta/\alpha_0}} \tag{6.39}$$

and we get the property

$${}^{\delta}\overline{B}^{\alpha}(t) \overset{d}{=} D_{\delta}^{1/\alpha} \cdot t^{\delta/\alpha} \cdot \left(\frac{S_{\alpha/2}}{S_{\delta}^{2\delta/\alpha}}\right)^{1/2} \cdot G. \tag{6.40}$$

From Bayes' formula, with $f_{{}^{\delta}\overline{S}^{\alpha_0}(t)}(s)$ the density of ${}^{\delta}\overline{S}^{\alpha_0}(t)$ at $s > 0$, if ${}^{\delta}\overline{\Phi}^{\alpha}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\cdot{}^{\delta}\overline{B}^{\alpha}(t)}$, we get

$${}^{\delta}\overline{\Phi}^{\alpha}(t,\lambda) = \boldsymbol{E}\mathrm{e}^{-q\cdot{}^{\delta}\overline{S}^{\alpha_0}(t)}|_{q=\lambda^2/2} = \phi_{\delta}\left(D_{\delta}t^{\delta}\left(\frac{\lambda^2}{2}\right)^{\alpha/2}\right) \tag{6.41}$$

in terms of the standard Mittag–Leffler function.

Thus, from (6.19), with $\delta \in (0,1)$ and $\alpha \in (0,2)$

$${}^{\delta}\overline{\Phi}^{\alpha}(p,\lambda) := \int_0^{+\infty} \mathrm{e}^{-pt}\boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\cdot{}^{\delta}\overline{B}^{\alpha}(t)}\,\mathrm{d}t = \frac{1}{p(1 + D_{\delta}p^{-\delta}(\frac{\lambda^2}{2})^{\alpha/2})}. \tag{6.42}$$

If $\alpha = 2$, we recover (6.11), as required.

From expression (6.41), the quantile dispersion grows as

$$\boldsymbol{D}[{}^{\delta}\overline{B}^{\alpha}(t)] \propto t^{\delta/\alpha}. \tag{6.43}$$

In (6.43), we see a competition between the 'rare event' hypothesis ($\delta < 1$) which tends to slow the dispersion of FT-LM and the 'extreme event' hypothesis ($\alpha < 2$) which tends to accelerate it. A particular case of interest is when $\delta = \alpha/2$ for which the critical diffusive regime is found; in this case, from (6.40): ${}^{\delta}\overline{B}^{2\delta}(t) \overset{d}{=} D_{\delta}^{1/(2\delta)} \cdot t^{1/2} \cdot S_{\delta,\delta}^{1/2} \cdot G$, with $S_{\delta,\delta}$ the random variable obtained as the ratio of two independent δ-Lévy-stable variables.

6.4.3. FT-FLM. In some sense, this is the most general of the processes studied in this paper as it encompasses all the ingredients introduced so far: fractal time (parameter δ), infinite memory (parameter H) and 'heavy tails' in space (parameter α).

Concerning the FT-FLM: ${}^{\delta}\overline{B}_H^{\alpha}(t) := \overline{B}_H^{\alpha}({}^{\delta}\overline{I}(t)) = \overline{B}_H({}^{\delta}\overline{S}^{\alpha/2}(t))$, with

$$\overline{B}_H(t) \overset{d}{=} t^H \cdot G \tag{6.44}$$

and

$${}^{\delta}\overline{S}^{\alpha/2}(t) \overset{d}{=} D_{\delta}^{2/\alpha} \cdot t^{2\delta/\alpha} \cdot \frac{S_{\alpha/2}}{S_{\delta}^{2\delta/\alpha}} \tag{6.45}$$

we get the distributional characterization

$${}^{\delta}\overline{B}_H^{\alpha}(t) \overset{d}{=} D_{\delta}^{2H/\alpha} \cdot t^{(2\delta H)/\alpha} \cdot \left[\frac{S_{\alpha/2}}{S_{\delta}^{2\delta/\alpha}}\right]^H \cdot G. \tag{6.46}$$

From Bayes' formula, setting ${}^{\delta}\overline{\Phi}_H^{\alpha}(t,\lambda) := \boldsymbol{E}\mathrm{e}^{\mathrm{i}\lambda\cdot{}^{\delta}\overline{B}_H^{\alpha}(t)}$, and after some algebraic manipulations similar to those in ([8], vol 2, XIII. 8, (8.4), p 453), we derive the generalized Mittag–Leffler formula

$${}^{\delta}\overline{\Phi}_H^{\alpha}(t,\lambda) = \zeta_{\delta,\alpha,H}\left(D_{\delta}^{2H} t^{2\delta H}\left(\frac{\lambda^2}{2}\right)^{\alpha/2}\right) \tag{6.47}$$

where the function $\zeta_{\delta,\alpha,H}(x)$, $x \geqslant 0$ is defined in the following way:

$$\zeta_{\delta,\alpha,H}(x) := \phi_{\delta,H}(-\log\tilde{\psi}_{\alpha,H}(x)). \tag{6.48}$$

Here the function $\phi_{\delta,H}(x)$ is the generalized Mittag–Leffler function defined in (6.30) and the function $\tilde{\psi}_{\alpha,H}(x)$ is defined by

$$\tilde{\psi}_{\alpha,H}(x) := \psi_{\alpha,H}(x^{2/\alpha}) \tag{6.49}$$

in terms of the function $\psi_{\alpha,H}(x)$ defined in (4.4).

We recall that $\psi_{\alpha,H}(x) = \boldsymbol{E}\mathrm{e}^{-xS_{\alpha/2}^{2H}}$, $x \geqslant 0$ is the Laplace transform of a $\alpha/2$-Lévy-stable variable raised to the power $2H$.

From (4.5), with $\alpha \in (0,2)$ and $H \in (0,1)$

$$\tilde{\psi}_{\alpha,H}(x) = 1 + \sum_{n\geqslant 1}\frac{(-1)^n}{n!}\frac{\Gamma(1-(n\alpha)/(4H))}{\Gamma(1-(n\alpha)/2)}x^{n/(2H)}. \tag{6.50}$$

Note that the formulae (6.47)–(6.49) completely characterize the distribution of FT-FLM; they are, of course, consistent with our previous findings:

(1) If $\delta = 1$, from (6.30) we have the identity $\phi_{1,H}(x) = \exp -x$. Thus, in this case, $\zeta_{1,\alpha,H}(x) = \tilde{\psi}_{\alpha,H}(x) = \psi_{\alpha,H}(x^{2/\alpha})$; formula (6.47) reduces to (4.4), relative to FLM, as required.

(2) If $H = \frac{1}{2}$, we have the identities: $\tilde{\psi}_{\alpha,1/2}(x) = \boldsymbol{E}\mathrm{e}^{-x^{2/\alpha}S_{\alpha/2}} = \mathrm{e}^{-(x^{2/\alpha})^{\alpha/2}} = \mathrm{e}^{-x}$ and $\phi_{\delta,1/2}(x) = \phi_{\delta}(x)$. Thus, in this case, $\zeta_{\delta,\alpha,1/2}(x) = \phi_{\delta}(x)$; formula (6.47) reduces to (6.41), relative to FT-LM, as required.

(3) If $\alpha = 2$, the random variable $S_{\alpha/2}$ is degenerate (and equal to one); thus, $\tilde{\psi}_{2,H}(x) = \mathrm{e}^{-x}$ and $\zeta_{\delta,2,H}(x) = \phi_{\delta,H}(x)$. We recover (6.29), relative to FT-FBM, as required.

(4) If both $\alpha = 2$ and $H = \frac{1}{2}$, $\zeta_{\delta,2,1/2}(x) = \phi_{\delta}(x)$ we recover (6.11), as required.

If $\alpha \neq 2$, from the expression (6.47), it is easy to establish that, for some appropriate constant $K > 0$,

$$^{\delta}\overline{\Phi}_H^{\alpha}(t, \lambda) \underset{\lambda \uparrow 0^+}{\sim} 1 - Kt^{\delta}|\lambda|^{\alpha/(2H)} \tag{6.51}$$

so that the tail behaviour follows:

$$\boldsymbol{P}(|^{\delta}\overline{B}_H^{\alpha}(t)| > x) \underset{x \uparrow +\infty}{\propto} t^{\delta} \cdot x^{-\alpha/(2H)}. \tag{6.52}$$

Consequently, the quantile dispersion grows as

$$\boldsymbol{D}[^{\delta}\overline{B}_H^{\alpha}(t)] \underset{t \uparrow +\infty}{\propto} t^{(2\delta H)/\alpha} \tag{6.53}$$

which is consistent with all the results (4.9), (6.31), (6.42), (6.12) presented so far. Large values of δ and H increase the dispersion of $^{\delta}\overline{B}_H^{\alpha}(t)$ whereas small values of α go in the same direction.

Remark 7. For the class of FT motions whose properties have just been discussed, one may wish to study the increment process

$$^{\delta}B_H^{\alpha}(t) := {^{\delta}\overline{B}_H^{\alpha}}(t+1) - {^{\delta}\overline{B}_H^{\alpha}}(t).$$

However, it is not true that this process is stationary: the identity $^{\delta}B_H^{\alpha}(t) \overset{d}{=} {^{\delta}B_H^{\alpha}}(0) = {^{\delta}\overline{B}_H^{\alpha}}(1)$ does not hold, chiefly because the subordinator $^{\delta}\overline{I}(t)$ itself has no SIs.

A fortiori, the property $(^{\delta}B_H^{\alpha}(s), {^{\delta}B_H^{\alpha}}(s+t)) \overset{d}{=} (^{\delta}B_H^{\alpha}(0), {^{\delta}B_H^{\alpha}}(t))$ is invalid and the covariance function depends on both s and t. Thus, for FT-FLM, in sharp contrast to FLM, the dependence structure of the increments cannot be understood using second-order statistics. Rather, the sample autocovariance

$$\lim_{s \uparrow +\infty} 1/s \int_0^{+s} {^{\delta}B_H^{\alpha}}(s) \cdot {^{\delta}B_H^{\alpha}}(s+t)\, \mathrm{d}s$$

should be used instead. However, in the limit, this quantity is random, owing to the lack of ergodicity of $^{\delta}B_H^{\alpha}(t)$.

Remark 8. With, a and $\sigma > 0$ some constants, one may define and study the FT version of multiplicative FLM, following section 5.2:

$$^{\delta}\overline{P}_H^{\alpha}(t) := \exp(at + \sigma \cdot {^{\delta}\overline{B}_H^{\alpha}}(t)) > 0.$$

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